Istituto Lombardo- Accademia di Scienze e Lettere
Giornata di Studio in memoria di Luigi Amerio

PHYSICS and MATHEMATICS MEET on the GROUND of UNCERTAINTY

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Abstract

The presentation aims at illustrating the intimate connections between Physics and Mathematics from the viewpoint of a theoretical physicist.
The first part discusses the feedback cycle which governs the evolution of the field of Physics. The second focuses on the theme of uncertainty, a fundamental connotation of human reality.

1. Introduction.

It is a great pleasure for me to present this talk on the occasion of this day dedicated to the memory of Luigi Amerio. When I was a student of the fourth year at the Master course in Physics at University of Milan , Professor Luigi Amerio delivered a course of lectures on the theme of Distributions. This course was not in my study plan, but I started attending it for curiosity and I was quickly captured by his teaching style, characterized by an extraordinary clarity, so that I attended his lectures to their very end. Indeed, his teaching style remained imprinted in my mind as an ideal example to follow in my teaching activity.

In this talk I will try to outline my vision of the profound relations between Physics and Mathematics, as I perceive them as a theoretical physicist. My presentation will be simple and nontechnical, with the aim of being accessible also to educated laymen who are not familiar with Physics and with Mathematics.

My starting point is the common claim that Science is the domain of certainty. This is right only in a zeroth approximation, as I will argue in my talk. This is subdivided in two parts. The first illustrates the feedback loop which determines the evolution of Physics. The second part addresses the central theme of uncertainty, a paradigm which affects the whole reality of mankind.

2. The feedback cycle of Physics

Reality is complex, where the word complexity has both the commonplace meaning of complication and the modern meaning of richness of phenomena and of aspects. This implies that, when we address a general topic in a reduced time, it is unavoidable to introduce schematizations and simplifications and I would like to warn that we will follow this path in this talk.
We will subdivide the feedback loop, which governs the evolution of Physics, into four stages.

First stage.

The first stage (Fig.1) is purely **inductive.** One collects the results of a large number of laboratory experiments. In this stage we are just **in the domain of certainty**, because the experimental results are **reproducible**. The experimentalists records the physical systems and the detection apparatuses that are used in the experimental setting, records the results and makes all these data public, so that any other experimentalist in any part of the world can reproduce the experiment.

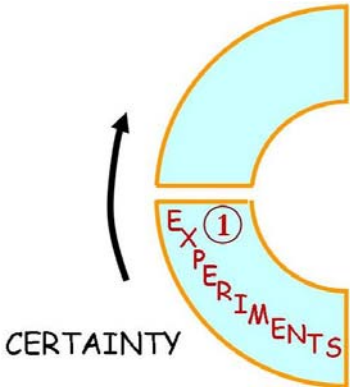

Fig.1. The first stage of the feedback cycle which governs the evolution of Physics.

Second stage

In the second stage there is a powerful joint effort of theoretical physicists and experimental physicists to **unify the experimental results and formulate a physical theory.**

It is at this stage that the meeting of Physics and Mathematics takes place, because Mathematics provides **the language and the instrument** to formulate the physical theory.

Since I would like to provide a precise idea about this meeting, I dwell now on the paradigmatic example of the fundamental law of dynamics or Newton's law, which reads

$$m\vec{a} = \vec{F} \qquad (1)$$

where *m* is the inertial mass of a particle, $\vec{a}$ is its acceleration and $\vec{F}$ is the force which acts on the particle itself. When I was a student in a classic highschool in Milan, I felt I was not able to grasp the real meaning of this law and especially the reason why it is named the fundamental law of dynamics. This point became completely clear to me in the first year of my Physics course at the University of Milan, because I learnt the mathematical analysis and, in the physics course, I learnt that the acceleration is the second order derivative of the position of the particle with respect to time, so that the law takes the form

$$m\frac{d^2\vec{x}}{dt^2} = \vec{F} \qquad (2)$$

where $\vec{x}$ denotes the position vector of the particle and t denotes time. In order to appreciate the meaning and the importance of this step it is not necessary to know what a second order derivative is . It is enough to know that, once written in the form (2), Newton's law becomes a differential equation, and the solutions of this equation are capable of describing, for example, the motion of a planet around the sun. Or, performing a dizzy jump in spatial scale from the astronomical scale to the microscopic scale as well as in time from Newton to Rutherford time, it can describe the motion of the electron around the proton in the hydrogen atom.

From all of this we can conclude that, rather than speaking of a meeting between Physics and Mathematics, it is more precise to speak of a twin birth which took place in the brain of personalities such as Newton and Leibniz.

If now we perform a further temporal leap to the first decades of the twentieth century, we can see that at that time the motion of the electron around the proton in the hydrogen atom is described by the more complex Schroedinger equation

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z)+V(x,y,z)\psi(x,y,z)=i\frac{h}{2\pi}\frac{\partial\psi(x,y,z)}{\partial t} \quad (3)$$

Where $\psi(x,y,z)$ is the wave function of the electron and one notes the sum of the three second order partial derivatives with respect to the three spatial coordinates *x*, *y* and *z* and the first order partial derivative with respect to time . The symbol h denotes the Planck's constant and *i* is the imaginary unit. The terms of the equation with second order derivatives are associated with the kinetic energy of the electron, while $V(x,y,z)$ is the potential energy associated with the electrostastic attraction force exerted by the proton.

I believe that it is easy to perceive that the Schroedinger equation, which is a partial differential equation, requires a more sophisticated mathematics than the Newton equation, which is an ordinary differential equation. The relevant remark is that the progress in Physics is intimately connected to that in Mathematics and that the marriage of the two continues uninterrupted with time.

But now let us come back to the second stage of the feedback loop (Fig.2). Here I must add the crucial observation that **now we are out of the domain of certainty**, because the physical theory can and must be subjected to newer and newer experimental tests. This is true even if without any doubt the physical theory is a powerful conceptual construction. Indeed, at the end of the nineteenth century it certainly appeared as a powerful building based on mechanics and thermodynamics on the one hand, and on electromagnetism on the other. However, this reality entered quickly into deep crisis which gave rise to the relativistic theory and to the quantum theory. Nobody can exclude that a similar story can repeat itself, of course nobody knows how and when.

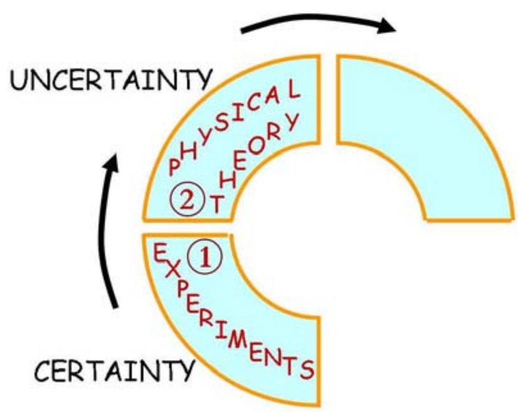

Fig.2 First and second stage of the feedback cycle which governs the evolution of Physics.

Third stage

The third stage is purely **deductive** and is operated by theoretical physicists. They start from the physical theory or from simplified versions of the physical theory that we will call **models.** Indeed the physical theory is quite complex, and to describe a certain phenomenon or class of phenomena it maybe enough to utilize some simple model in which some unnecessary ingredients are neglected and/or some suitable approximations are introduced..

By using the tools of Mathematical Analysis or/and Numerical Analysis (i.e. computers), theoretical physicists aim to **discover novel phenomena** (Fig.3). Possibly, these phenomena may be interesting not only from a fundamental viewpoint, but also in view of **future applications**.

Finally theoretical physicists publish the results of their researches and pass them to the hands of their experimental colleagues.

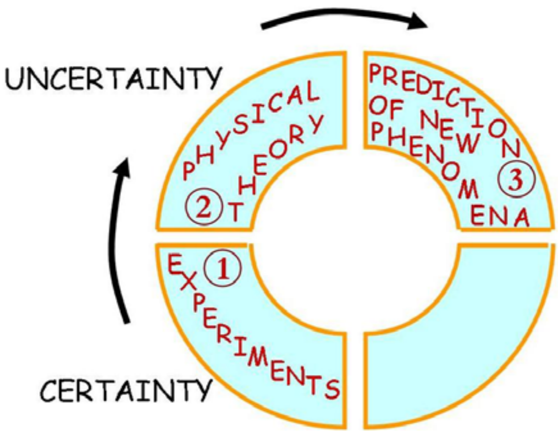

Fig.3. The first three stages of the feedback cycle which governs the evolution of Physics.

Fourth stage

It is important to take into account that in Physics, as well as in all Sciences with the exception of pure Mathematics, the theoretical predictions become really valid only when they are confirmed by experiments.
In the fourth stage the experimental physicists verify the theoretical predictions and here, obviously, two different scenarios are possible.

In case the results of the experiment are positive with respect to the theoretical predictions, one has the **ultimate validation of the new phenomenon**.
Possibly the researches on the new phenomenon pass in the hands of **engineers and technologists** for applicative developments.

In case the results of the experiment are negative with respect to the theoretical predictions, this indicates the presence of errors or defects in the model which has been utilized, in the sense that some ingredients which have been neglected in the derivation of the model from the physical theory are instead necessary, or some approximations which have been introduced in the derivation of the model are not adequate. In extreme cases, the incompatibility between experimental results and theoretical predictions points to the presence of **errors or defects in the physical theory itself**, which is therefore set in **crisis** with **revolutionary consequences.** This is what happened, for example, in the passage from classical to quantum Physics, as it was mentioned before.

The feedback cycle of Physics is now concluded (Fig.4). The experimental investigations induce novel theoretical researches and, conversely, the theoretical investigations prime novel experimental researches, and the two processes work **simultaneously** and **continuously**.

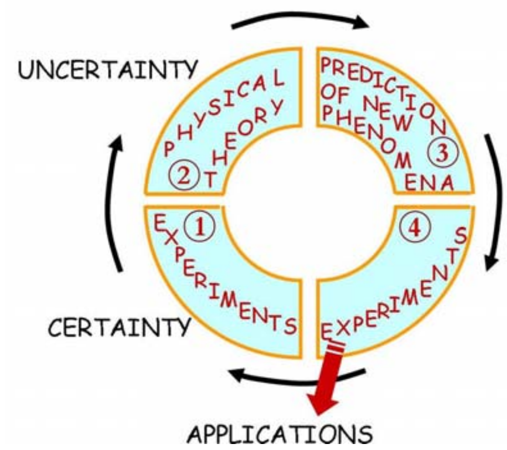

Fig 4. The entire feedback cycle which governs the evolution of Physics.

It must be underlined that there are also experimental researches which are purely curiosity driven and lead to the discovery of novel phenomena. In this case, the experimentalists pass their results to the theoreticians, who then investigate to provide a theoretical explanation of the new phenomena.

In the scheme of figure 4, applied physics appears in a secondary position with respect to pure physics. Indeed, this vision is not correct because pure and applied physics influence each other continuously, so that each of the two can progress only thanks to the presence of the other. For example, the invention of the telescope was essential for the spectacular progress in the field of mechanics which took place in the seventeenth century. And also at our times the pure physics researches for elementary particles and astrophysics can advance only thanks to the development of the most sophisticated and powerful technological realizations.

3. Uncertainty and randomness, a fundamental connotation of human reality

Mathematics and randomness

Due to the rigour of its demontrations, Mathematics is certainly the fortress of certainty. If we think, for example, of the demonstration of some complex theorem, in which after a tight sequence of logical steps one arrives at the conclusion of the " quod erat demonstrandum", one gets a sense of satisfaction and of reassurance, feeling so well on the ground of certainty.

On the other hand, Mathematics has been able to even **rationalize the uncertainty**, by introducing the concepts of **probability** and of **statistics**. The so called **stochastic equations** do not lead to a precise result, but to the **probability distribution** of obtaining a certain result among all the results which are a priori possible.

Physics and randomness

In the case of Physics, the existence of randomness emerges from the **fluctuations** that we observe in the experimental measurements. Let we consider, for example, the intensity of the light emitted by an ordinary source such as the sun or a lamp. If we use a fast detector we find large fluctuations around the mean value (Fig. 5a) ; fortunately our eyes is not capable of detecting them because of its slow response times. In the case of a laser light source the fluctuations of the light intensity are much smaller and this feature is one of the properties which define the coherence of the laser (Fig.2b)..

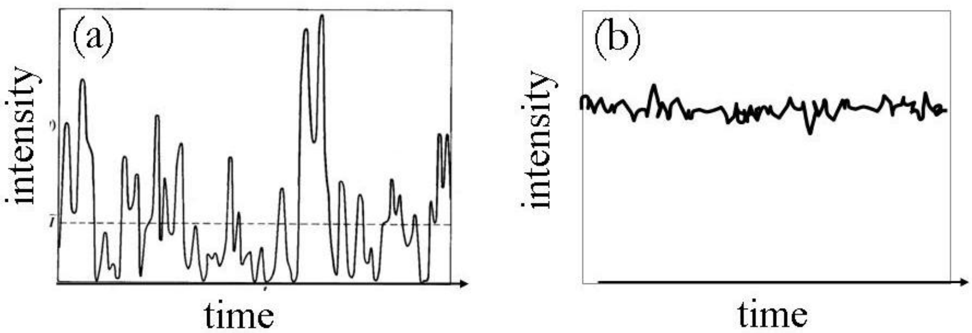

Fig.5. (a) Fluctuations in the intensity of the light emitted by an ordinary source. (b) Fluctuations in the intensity of the light emitted by a laser source.

Fluctuations arise from the circumstance that the experimentalist is capable of exerting only a **limited control** on the conditions of the physical system. He can try to fix all parameters of the system as precisely as possible, but only to a certain extent. He cannot, for example, control the positions and the velocities of all particles which constitute the system..

In this connection it is worth noting that also in our ordinary life randomness arises from the limited control we can exert on the environment in which we live.

Also in this case the meeting between Physics and Mathematics occurs **on the ground of uncertainty** and also in this case it has been fundamental and extremely fertile. For example, the use of the concepts of probability and statistics has allowed the creation of the discipline of Statistical Mechanics, which has provided the microscopic foundation of thermodynamics, one of the columns of Physics and Chemistry.

The role of randomness in Physics, and more in general in Science, has been brought to extreme consequences by a **revolution** occurred in the first decades of the twentieth century and by a **surprising discovery** occurred in the second part of that century. The revolution was the coming of quantum physics as we already mentioned a couple of times. The surprising discovery was that of **chaos** in nonlinear dynamical systems.

Quantum Physics

Quantum theory is **intrinsically statistical**. In the case of microscopic systems, the state of the system is expressed purely and exclusively **in terms of probabilities**, at difference from the equations of classical theory which are **deterministic**. For example, the modulus squared of the complex wave function ψ that obeys the Schroedinger equation is the **probability distribution** of finding the electron in the points of the space around the proton.

In the case of macroscopic systems the quantum picture, with the associated statistical description, is strictly necessary only in the case of systems at ultralow temperatures close to the absolute zero. In the other situations the classical picture usually provides an adequate approximation.

Deterministic chaos

This phenomenon arises in all fields of Science (Physics, Chemistry, Natural and Environmental Sciences, Social and Economical Sciences etc.) in the framework of nonlinear dynamical systems, i.e. in systems in which the effects are not proportional to the causes.

Chaos is called **deterministic** because it arises in deterministic equations, which do not include any source of fluctuations, contrary to stochastic equations. In the case of deterministic equations, once the **initial state** of the system is assigned, the following **temporal evolution** of the state itself is univocally determined.

However, chaos is a sort of **intrinsic unpredictability** and therefore one wonders how this can happen in a deterministic equation. T
In order to understand this the key point is to take into account that the initial state can be determined only within a certain uncertainty, which may be small or even very small but can never be zero. In nonlinear equations, under standard conditions this initial error can grow in time but in a controllable way, for example linearly with time. In the case of chaos, instead, the initial error **grows exponentially**, i.e. extremely rapidly with time, with the consequence that the prediction of the temporal evolution becomes quickly unreliable. Think, for example, of the weather forecasts, which are reliable only for a few days.

In a sense deterministic chaos provides the scientific explanation for the presence of uncertainty (unpredictability) in the reality which surrounds us. If one assumes a very pragmatic and a little bit rogue attitude, one might observe that we knew already about this characteristic of reality. On the other hand, another fact that we know already very well is that we are capable of surviving even in this uncertain reality. And maybe this is the only certainty which counts

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