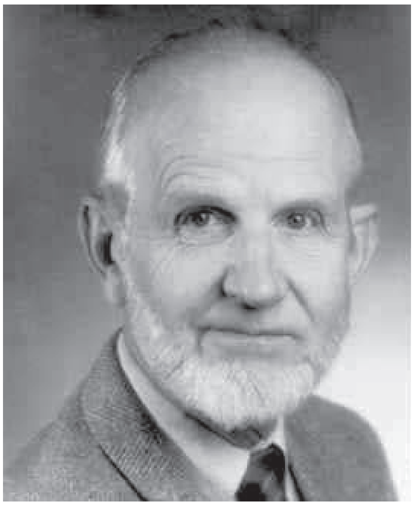
A tribute to Lars Hörmander

par Nicolas Lerner [1]



LARS HÖRMANDER, 1931–2012

Foreword

Lars Hörmander died on November 25, 2012, at the age of 81. He was one of the most influential mathematicians of the twentieth century. He played a fundamental role in the development of the analysis of partial differential equations for more than forty years, displaying exceptional technical abilities combined with a broad and deep vision of the subject. His style of exposition was characterized by concision, precision and completeness.
He was awarded the Fields Medal in 1962, the Wolf Prize in 1988, and the Steele Prize in 2006. His monumental four-volume treatise, *The Analysis of Linear Partial Differential Operators*, is considered to be the ultimate reference on the topic of linear partial differential operators. He was a member of the Swedish Royal Academy since 1968, was elected as a member of the USA National Academy of Sciences in 1976 and served between 1987 and 1990 as a vice-president of the International Mathematical Union.

[1]N. Lerner, Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie (Paris VI), 4 Place Jussieu, 75252 Paris cedex 05, France -lerner@math.jussieu.fr

Before the Fields Medal

Lars Hörmander was born in 1931, in southern Sweden, where his father was a teacher. He got his high-school degree in 1948 and a master's degree two years later at the age of nineteen at the University of Lund, with M. Riesz as an advisor. He wrote a Ph.D. thesis under the guidance of L. Gårding and the publication of that thesis, *On the theory of general partial differential operators* [1] in *Acta Mathematica* in 1955 can be considered as the starting point of a new era for Partial Differential Equations.
Among other things, very general theorems of local existence were established, without using an analyticity hypothesis of the coefficients. L. Hörmander's arguments relied on a priori inequalities combined with abstract functional analytic arguments. Let us cite L. Gårding in [2], writing about a general linear PDE

$$P(x, D_x)u = f. \tag{1}$$

> *It was pointed out very emphatically by Hadamard that it is not natural to consider only analytic solutions and source functions f even if P has analytic coefficients. This reduces the interest of the Cauchy-Kowalevski theorem which says that* (1) *has locally analytic solutions if P and f are analytic. The Cauchy-Kowalevski theorem does not distinguish between classes of differential operators which have, in fact, very different properties such as the Laplace operator and the Wave operator.*

L. Hörmander's filiation with J. Hadamard's work is clear. J. Hadamard (1865–1963) introduced the fruitful notion of well-posedness for a PDE problem : existence, uniqueness are important properties, but above all, continuous dependence of the solution with respect to the data should be emphasized as one of the most important properties for a PDE. After all, the data (boundary or Cauchy data, various quantities occurring in the equation) in a Physics problem are known only approximately and even if the solution existed and was proven unique, this would be useless for actual computation or applications if minute changes of the data triggered huge changes in the solution. In fact, one should try to establish some *inequalities* controlling the size of the norms or semi-norms of the solution u in some functional space. The lack of well-posedness is linked to instability and is also a very interesting phenomenon to study. We can quote again at this point L. Gårding (op.cit.) :

> *When a problem about partial differential operators has been fitted into the abstract theory, all that remains is usually to prove a suitable*

inequality and much of our knowledge is, in fact, essentially contained in such inequalities.

L. Ehrenpreis [3] and B. Malgrange [4] had proven a general theorem on the existence of a fundamental solution for any constant coefficients PDE, and the work [5] by L. Hörmander provided another proof along with some improvement on the regularity properties, whereas [1] gave a characterization of hypoelliptic constant coefficients PDE, in terms of properties of the algebraic variety

$$\text{char}P = \{\zeta \in \mathbb{C}^n, P(\zeta) = 0\}.$$

The operator $P(D)$ is hypoelliptic if and only if

$$|\zeta| \to \infty \text{ on char}P \Longrightarrow |\operatorname{Im}\zeta| \to \infty.$$

Here hypoellipticity means $Pu \in C^\infty \Longrightarrow u \in C^\infty$. The characterization of hypoellipticity of the constant coefficient operator $P(D)$ by a simple algebraic property of the characteristic set is a tour de force, technically and conceptually : in the first place, nobody had conjectured such a result, or even remotely suggested a link between the two properties and next, the proof provided by L. Hörmander relies on a very subtle study of the characteristic set, requiring an extensive knowledge of real algebraic geometry.

In 1957, Hans Lewy made a stunning discovery [6] : the equation $\mathcal{L}u = f$ with

$$\mathcal{L} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3} \tag{2}$$

does not have local solutions for most right-hand-sides f. The surprise came in particular from the fact that the operator $\mathcal{L}$ is a non-singular (i.e. non-vanishing) vector field with a very simple expression and also, as the Cauchy-Riemann operator on the boundary of a pseudo-convex domain, it is not a cooked-up example. L. Hörmander started working on the Lewy operator (2) with the goal to get a general geometric understanding of a class of operators displaying the same defect of local solvability. The two papers [7], [8], published in 1960 achieved that goal. Taking a complex-valued homogeneous symbol $p(x, \xi)$, the existence of a point (x, ξ) in the cotangent bundle such that

$$p(x, \xi) = 0, \quad \{\bar{p}, p\}(x, \xi) \neq 0 \tag{3}$$

ruins local solvability at x (here $\{\cdot, \cdot\}$ stands for the Poisson bracket). With this result, L. Hörmander gave a generalization of Lewy operator and provided a geometric explanation in invariant terms of that non-solvability phenomenon. We may note also that Condition (3) is somehow generically satisfied : considering a

non-elliptic operator with a complex-valued principal symbol p, the symbol p will vanish somewhere and generically $\{\bar{p}, p\} \neq 0$ there, so that "most" non-elliptic operators with a complex-valued symbol are non-solvable.
A. Calderón's 1958 paper [9] on the uniqueness in the Cauchy problem was somehow the starting point for the renewal of singular integrals methods in local analysis. Calderón proved in [9] that an operator with real principal symbol with simple characteristics has the Cauchy uniqueness property ; his method relied on a pseudodifferential factorization of the operator which can be handled thanks to the simple characteristic assumption. It appears somewhat paradoxical that L. Hörmander, who became later one of the architects of pseudodifferential analysis, found a generalization of Calderón's paper using only a local method, inventing a new notion to prove a Carleman estimate. He introduced in [10], [11] the notion of pseudo-convexity of a hypersurface with respect to an operator, and was able to handle the case of tangent characteristics of order two.
In 1957, L. Hörmander was appointed Professor at the University of Stockholm, where he was to stay until 1964, but he also spent some time in Stanford University as well as at the Institute for Advanced Study in Princeton.
In 1962, at the age of 31, L. Hörmander was awarded the Fields Medal. His impressive work on Partial Differential Equations, in particular his characterization of hypoellipticity for constant coefficients and his geometrical explanation of the Lewy non-solvability phenomenon were certainly very strong arguments for awarding him the Medal. Also his new point of view on PDE, which combined functional analysis with a priori inequalities, had led to very general results on large classes of equations, which had been out of reach in the early fifties. L. Hörmander wrote in the book *Fields Medallists' lectures* [12] :

> *The 1962 ICM was held in Stockholm. In view of the small number of professors in Sweden at the time, it was inevitable that I should be rather heavily involved in the preparations, but it came as a complete surprise to me when I was informed that I would receive one of the Fields medals at the congress.*

From the first PDE book to the four-volume treatise

L. Hörmander spent the summers 1960-61 at Stanford University as an invited professor, and took advantage of this time to honour the offer of the *Springer Grundlehren series* of publishing a book about PDE. It was done in 1963, with the publication of his first book, *Linear partial differential operators.* That book was a milestone in the study of PDE, and a large mathematical public discovered L. Hör-

mander's exposition of recent progress in the area.
In the first place, the rôle of Distribution Theory was emphasized as the perfect tool for linear PDE. Although the notion of weak solution for a PDE was already known to S. Sobolev and to the Russian school in the thirties, it is indeed L. Schwartz' definition of distributions which created the best perspective, combining abstract aspects of functional analysis with Fourier analysis. L. Hörmander had been familiar for quite a long time with Schwartz theory, but he had noticed that many mathematicians, including his mentor M. Riesz, were rather negative (to say the least) about it. F. Treves in [13] tells the following anecdote : L. Schwartz visited Lund University in 1948 and gave a talk there on some elements of distribution theory. Having written on the board the integration by parts formula to explain the idea of a weak derivative, he was interrupted by M. Riesz saying "I hope you have found something else in your life." Also, M. Riesz claimed that the known examples of fundamental solutions of hypoelliptic PDE with constant coefficients were always locally integrable, so that distributions were useless for their study. A few years after his retirement, Hörmander went back to this problem and found an example in 14 dimensions of an hypoelliptic operator whose fundamental solution is not locally integrable (see [14]). During his thesis work, L. Hörmander managed to avoid explicit reference to Schwartz theory, but in 1963, it was a different story and he chose to present Schwartz Distribution Theory as the basic functional analytic framework of his book. As a matter of fact, the first chapter of his book is devoted to a (dense) presentation of the theory, including the geometric version on manifolds.
Large segments of the book were devoted to constant coefficient operators, but it contained also a great deal of the recent progress on uniqueness for the Cauchy problem, with Carleman estimates and a wide array of counterexamples due to A. Pliś and P. Cohen. Anyhow, that book became soon a standard text which must be studied by anybody wishing to enter the PDE field.
L. Hörmander's career developed at a quick pace after the Fields medal ; he wrote in [15] :

> *Some time after two summers (1960, 1961) at Stanford, I received an offer of a part time appointment as professor at Stanford University...I had barely arrived at Stanford when I received an offer to come to the Institute for Advanced Study as permanent member and professor. Although I had previously been determined not to leave Sweden, the opportunity to do research full time in a mathematically very active environment was hard to resist...I decided in the fall of 1963 to accept the offer from the IAS and resign from the universities of Stockholm and Stanford to take up a new position in Princeton in the fall of 1964.*

Hypoellipticity

A. Kolmogorov introduced in 1934 the operator in $\mathbb{R}^3_{t,x,v}$

$$\mathcal{K} = \partial_t + v\partial_x - \partial_v^2, \tag{4}$$

to provide a model for Brownian motion in one dimension. That was L. Hörmander's starting point. He took up the study of general operators

$$\mathcal{H} = X_0 - \sum_{1 \le j \le r} X_j^2, \tag{5}$$

where the $(X_j)_{0 \le j \le r}$ are smooth real vector fields whose Lie algebra generates the tangent space at each point. The rank of the X_j and their iterated Poisson brackets is equal to the dimension of the ambient space (for $\mathcal{K}$, we have $X_0 = \partial_t + v\partial_x, X_1 = \partial_v, [X_1, X_0] = \partial_x$). These operators were proven in [16] to be hypoelliptic, i.e. such that $\operatorname{singsupp} u = \operatorname{singsupp} \mathcal{H}u$ for the C^∞ singular support. This Hörmander paper was the starting point of many studies, including numerous articles in probability theory and the operators $\mathcal{H}$ soon became known as *Hörmander's sum of squares*. Their importance in probability came from the fact that these operators appeared as a generalization of the heat equation where the diffusion term $\sum_{1 \le j \le r} X_j^2$, was no longer elliptic, but had instead some hypoelliptic behavior.

Pseudodifferential Equations

The aforementioned article by A. Calderón on uniqueness for the Cauchy problem led to renewed interest in singular integrals and the notion of pseudodifferential operator along with a symbolic calculus was introduced in the sixties by several authors : J.J. Kohn and L. Nirenberg in [17], A. Unterberger and J. Bokobza in [18]. L. Hörmander wrote in 1965 a synthetic account of the nascent pseudodifferential methods with the article [19].

Complex analysis

The now classical book, *An introduction to complex analysis in several variables* [20] and the paper [21] provide a PDE point of view on the holomorphic functions of several variables : they are considered as solutions of a PDE, the $\overline{\partial}$ system, and that perspective along with L^2 estimates turned out to be very fruitful for their study. Here is an excerpt from the preface of the book :

> *Two recent developments in the theory of partial differential equations have caused this book to be written. One is the theory of overdetermined systems of differential equations with constant coefficients,*

> *which depends very heavily on the theory of functions of several complex variables. The other is the solution of the so-called $\overline{\partial}$ Neumann problem, which has made possible a new approach to complex analysis through methods from the theory of partial differential equations. Solving the Cousin problems with such methods gives automatically certain bounds for the solution, which are not easily obtained with the classical methods, and results of this type are important for the applications to overdetermined systems of differential equations.*

Inhomogeneous Cauchy-Riemann equations in a polydisc, power series, Reinhardt domains, domains of holomorphy, pseudo-convexity and plurisubharmonicity, and Runge domains are dealt with in the second chapter. Included are theorems due to Hartogs, Dolbeault-Grothendieck, Cartan [22], Cartan-Thullen [23], Bochner [24], Lewy [25], Oka [26], Serre [27], and Browder [28]. After a chapter on commutative Banach algebras, Chapter IV is devoted to existence and approximation theorems for solutions of the inhomogeneous Cauchy-Riemann equations in domains of holomorphy. The technique is to prove L^2 estimates involving weight functions. Next, L. Hörmander introduces the notion of Stein manifolds, which are modeled on the properties of domains of holomorphy in $\mathbb{C}^n$. The theorems on existence and approximations of solutions of the Cauchy-Riemann equations are extended to these manifolds and it is shown that a manifold is a Stein manifold if and only if it can be represented concretely as a closed submanifold of a space $\mathbb{C}^N$ of sufficiently high dimension. Analytic continuation and the Cousin problems are studied for Stein manifolds. These results are due to Cartan, Grauert, Bishop, Narasimhan, and Oka. Chapter VI gives the Weierstrass preparation theorem and studies divisibility properties in the ring A_0 of germs of analytic functions. Submodules of A_0^p are studied along with K. Oka's theorem on the module of relations [29]. This is needed for the theory of coherent analytic sheaves, which is presented in the next and final chapter. There the study of the Cousin problems is extended to coherent analytic sheaves on Stein manifolds. A discussion of the theorem of Siu [30] on the Lelong numbers of plurisubharmonic functions is added. The L^2 techniques are essential in the proof and plurisubharmonic functions play such an important rôle that it is natural to discuss their main singularities.

Spectral Asymptotics

The article [31], which contains the first occurrence of Fourier Integral Operators, provides the best possible estimates for the remainder term in the asymptotic formula for the spectral function of an arbitrary elliptic (pseudo)differential operator. This is achieved by means of a complete description of the singularities of the Fourier transform of the spectral function for low frequencies.

In spite of this outstanding activity, L. Hörmander did not feel that comfortable at the IAS :

> *It turned out that I found it hard to stand the demands on excellence that inevitably accompany the privilege of being an Institute professor. After two years of very hard work I felt that my results were not up to the level which could be expected. Doubting that I would be able to stand a lifetime of pressure, I started to toy with the idea of returning to Sweden when a regular professorship became vacant. An opportunity arose in 1967, and I decided to take it and return as professor in Lund from the fall term 1968. "*

So in 1968, L. Hörmander had completed a full circle and was back in Lund where he had started as an undergraduate in 1948. He was to remain there until his retirement, with interruptions for some visits, mainly in the US.

The microlocal revolution

The fact that singularities should be classified according to their spectrum was recognized first in the early seventies, by three Japanese mathematicians : the Lecture Notes [32] by M. Sato, T. Kawai and M. Kashiwara set the basis for the analysis in the phase space and microlocalization. The analytic wave-front-set was defined in algebraic terms and elliptic regularity as well as propagation theorems were proven in the analytic category. The paper [33] by J. Bros and D. Iagolnitzer gave a formulation of the analytic wave-front-set that was more friendly to analysts.
The definition of the C^∞ wave-front-set was given in Hörmander's [34] by means of pseudodifferential operators. The propagation-of-singularities theorem for real principal type operators (see e.g. Hörmander's [35]) represents certainly the apex of microlocal analysis. Since the seventeenth century with the works of Huygens and Newton, the mathematical formulation for propagation of linear waves lacked correct definitions. The wave-front-set provided the ideal framework : for P a real principal type operator with smooth coefficients (e.g. the wave equation) and u a function such that $Pu \in C^\infty$, WFu is invariant by the flow of the Hamiltonian vector field of the principal symbol of P. These results found new proofs via Hörmander's articles on Fourier Integral Operators [36] and [37] (joint work with J. Duistermaat). It is interesting to quote at this point the introduction of [36] (the reference numbers are those of our reference list) :

> *The work of Egorov is actually an application of ideas from Maslov [38] who stated at the International Congress in Nice that his book actually contains the ideas attributed here to Egorov [39] and Arnold [40] as well as a more general and precise operator calculus than ours. Since the book is highly inaccessible and does not appear to be quite rigorous we can only pass this information on to the reader, adding a reference to the explanations of Maslov's work given by Buslaev [41]. In this context we should also mention that the "Maslov index" which plays an essential role in Chapters III and IV was already considered quite explicitly by J. Keller [42]. It expresses the classical observation in geometrical optics that a phase shift of* $\pi/2$ *takes place at a caustic. The purpose of the present paper is not to extend the more or less formal methods used in geometrical optics but to extract from them a precise operator theory which can be applied to the theory of partial differential operators. In fact, we only use the simplest expansions which occur in geometrical optics, and a wealth of other ideas remain to be investigated.*

The introduction of the next article [37] begins with

> *The purpose of this paper is to give applications of the operator theory developed in the first part. These concern the existence and regularity of solutions of*
>
> $$Pu = f$$
>
> *in a manifold* X*. In particular we construct and study parametrices for* P*; we consider the above equation under the assumption that* P *has a principal symbol* p *which is homogeneous of degree* m *and real.*

Local Solvability

After Lewy's counterexample (2) and L. Hörmander's work on local solvability mentioned above, L. Nirenberg and F. Treves in 1970 ([43], [44], [45]), after a study of complex vector fields in [46] (see also the S.Mizohata paper [47]), introduced the so-called condition (Ψ), and provided strong arguments suggesting that this geometric condition should be equivalent to local solvability. The necessity of condition (Ψ) for local solvability of principal-type pseudodifferential equations was proved in two dimensions by R. Moyer in [48] and in general by L. Hörmander ([49]) in 1981.

The sufficiency of condition (Ψ) for local solvability of differential equations was proved by R. Beals and C. Fefferman ([50]) in 1973. They created a new type of pseudodifferential calculus, based on a Calderón-Zygmund decomposition, and were able to remove the analyticity assumption required by L. Nirenberg and F. Treves. The sufficiency of that geometric condition was proven in 1988 in two dimensions by N. Lerner's [51]. Much later in 1994, L. Hörmander, in his survey article [52], went back to local solvability questions giving a generalization of N.L.'s article [53]. In 2006, N. Dencker [54] proved that condition (Ψ) implies local solvability with loss of two derivatives.

More on pseudodifferential calculus

The outstanding results by R. Beals and C. Fefferman [50] on local solvability of differential equations were supplemented by L. Hörmander's paper [55] in which a propagation argument provides local existence of C^∞ solutions for C^∞ right-hand-sides. However, a most striking fact in R. Beals and C. Fefferman's proof was the essential use of a non-homogeneous pseudodifferential calculus which allowed a finer microlocalization than what could be given by conic microlocalization. The efficiency and refinement of the pseudodifferential machinery was such that the very structure of this tool attracted the attention of several mathematicians, among them R. Beals and C. Fefferman [56], R. Beals [57], A. Unterberger [58]. L. Hörmander's 1979 paper [59], *The Weyl calculus of pseudodifferential operators*, represents an excellent synthesis of the main requirements for a pseudo-differential calculus to satisfy ; that article was used by many authors in multiple circumstances and the combination of the symplectically invariant Weyl quantization along with the datum of a metric on the phase space was proven to be a very efficient approach.

Writing the four-volume book, 1979-1984

On March 25, 1982, L. Hörmander received a *Doctorate Honoris Causa* from the Université Paris-sud at Orsay. The main scientific address was written by J.-M. Bony and J. Sjöstrand. The whole PDE community in Orsay and elsewhere was waiting for Hörmander's forthcoming book to appear in the *Springer Grundlehren series*. Three or four volumes, joint work or not, table of contents, nothing was clear-cut at this moment and the expectations were high that the book would represent a landmark in the history of PDE. The first two volumes appeared in 1983.

First volume : Distribution Theory and Fourier Analysis

It is now a classical book of Analysis and an excellent presentation of Distribution Theory. In particular, that introduction remains elementary and free from very abstract functional analytic arguments. In the notes of Chapter II, L. Hörmander is wrote :

> *The topology in $C_0^\infty(X)$ is the inductive limit of the topology in $C_0^\infty(K)$ when the compact set K increases to X, so it is a LF topology. We have avoided this terminology in order not to encourage the once current misconception that familiarity with LF space is essential for the understanding of distribution theory.*

As a result, this first volume is highly readable and represents a useful tool for teaching various elements of distribution theory. The organization of the whole treatise is also quite impressive : for instance Chapter I in this first volume contains a quite refined notion of partitions of unity, not to be used before Chapter XVIII in the third volume. Several mathematical gems can be found in this first volume : a new proof of the Schwartz kernel theorem in Chapter V, a proof of the Malgrange preparation theorem, an extensive study of the methods of stationary phase in Chapter VII. Self-containedness is also perfect : the very classical Gaussian integrals get computed explicitely, the three-page treatment of the Airy function in Chapter 7 is a model of concision and clarity.

Second volume : Differential Operators with Constant Coefficients

L. Hörmander writes in the preface to this volume :

> *This volume is an expanded version of Chapters III, IV, V and VII of my 1963 book…The main technical tool in this volume is the Fourier-Laplace transformation. More powerful methods for the study of operators with variable coefficients will be developed in Volume III. However, the constant coefficient theory has given the guidelines for all that work. Although the field is no longer very active - perhaps because of its advanced state of development -…the material presented here should not be neglected by the serious student who wants to get a balanced perspective of the theory…*

The third and fourth volumes appeared two years later in 1985. L. Hörmander is writes in the preface to these volumes :

> *The first two volumes of this monograph can be regarded as an expansion of my book…published in the Grundlehren series in 1963. However, volumes III and IV are almost entirely new. In fact they are*

> *mainly devoted to the theory of linear differential operators as it has developed after 1963. Thus the main topics are pseudodifferential and Fourier integral operators with the underlying symplectic geometry.*

Here the style of writing has drastically changed : these last two volumes are no longer intended for gifted graduate students, but the targeted readership is obviously researchers, already conversant with some technicalities of the subject.

Third volume : Pseudodifferential operators

Chapter XVII may be an exception to the above remark, although the technique of Carleman estimates is far form easy, the content of that chapter remains elementary as far as tools are concerned.
Chapter XVIII is concerned with pseudodifferential calculus : the 30-page presentation of the *Basic Calculus* is certainly an excellent introduction to the topic and L. Hörmander was cautious enough to give a separated treatment of the most classical case of pseudodifferential calculus, leaving aside the refinements for later sections in the same chapter. R. Melrose's totally characteristic calculus ([60]) and L. Boutet de Monvel's transmission condition ([61]) are given a detailed treatment in this chapter. The last sections are devoted to the Weyl calculus as described in L. Hörmander [59] and results on new lower bounds by C. Fefferman and D.H. Phong [62] are also given a thorough treatment.
Chapter XIX deals with elliptic operators on a manifold without boundary and the index theorem. In the Notes of Chapter XVIII, L. Hörmander writes :

> *It seems likely that it was the solution by Atiyah and Singer [63] of the index problem for elliptic operators which led to the revitalization of the theory of singular integral operators.*

Chapter XX is entitled *Boundary Problems for Elliptic Differential Operators*. It reproduces at the beginning elements of Chapter X in [64] and takes into account the developments on the index problem for elliptic boundary problems given by L. Boutet de Monvel [61], [65] and G. Grubb [66].
Chapter XXI is a presentation of symplectic geometry and begins with a series of classical results. Next, one finds various sharp results on normal forms of smooth functions in a symplectic space, in particular the results of J. Duistermaat and J. Sjöstrand [67]. Also this chapter is an important preparation for local solvability results of Chapter XXVI with the normal form given in the paper by L. Nirenberg and F. Treves [44]. Section 21.5 is devoted to the symplectic reduction of complex-valued quadratic forms and remains an excellent reference on the topic.
Chapter XXII is concerned with hypoelliptic operators : on the one hand, operators with a pseudodifferential parametrix, such as the hypoelliptic constant

coefficient operators and on the other hand generalizations of the Kolmogorov operators (5). Results on lower bounds for pseudodifferential operators due to A. Melin [68] are a key tool in this analysis. Results of L. Boutet de Monvel [69], J. Sjöstrand [70], L. Boutet de Monvel, A. Grigis and B. Helffer [71] are given.

Chapter XXIII deals with the classical topic of strictly hyperbolic equations and begins with the exposition of the classical energy method. The classical estimates are obtained for first order pseudodifferential operators and then a factorization argument allows one to deal with higher order operators. Also a version of the Lax-Mizohata theorem is given, which asserts the necessity of weak hyperbolicity for a weak version of well-posedness, following the work by V. Ivrii and V. Petkov [72].

The last chapter in volume 3 is Chapter XXIV, which is devoted to the mixed Dirichlet-Cauchy problem for second order operators. Singularities of solutions of the Dirichlet problem arriving at the boundary on a transversal bicharacteristic will leave again on the reflected bicharacteristic. The study of tangential bicharacteristics required a new analysis and attracted the attention of many mathematicians. Among these works : the papers by R. Melrose [73], M. Taylor [74], G. Eskin [75], V. Ivrii [76], R. Melrose and J. Sjöstrand [77], [78], K. Andersson and R. Melrose [79], J. Ralston [80], J. Sjöstrand [81].

Volume 3 should not be left without paying attention to the two appendices, providing a self-contained description of classical results on distributions in an open manifold as well as the exposition of some tools of differential geometry.

Fourth volume, Fourier integral operators

Chapter XXV is devoted to the theory of Fourier integral operators, including the case of complex phase. Although the propagation-of-singularities theorem for real principal type operators is already proven by pseudodifferential methods in a previous chapter (XXIII), the FIO method provides another proof.

Chapter XXVI deals with principal type operators. The real principal type case appears now as quite simple and the second section drives us into the much more complicated realm of complex-valued symbols. The necessity of condition (Ψ) for local solvability, taken from the already mentioned [49] and [48] is proven in Section 26.4. The last seven sections of this chapter are devoted to very precise propagation theorems for operators with complex symbols satisfying the stronger condition (P). The main ingredients used in the proof are the Malgrange preparation theorem, Egorov's theorem on conjugation of pseudodifferential operators by Fourier integral operators, Nirenberg-Treves estimates on degenerate Cauchy-Riemann equations [44], Beals-Fefferman non-homogeneous localization

procedure [50] and Hörmander's propagation result [55].
Chapter XXVII is concerned with subelliptic operators. A pseudodifferential operator of order m is said to be subelliptic with a loss of δ derivatives whenever

$$Pu \in H^s_{loc} \Longrightarrow u \in H^{s+m-\delta}_{loc}. \tag{6}$$

The elliptic case corresponds to $\delta = 0$, whereas the cases $\delta \in (0, 1)$ are much more complicated to handle. The first complete proof for operators satisfying condition (P) was given by F. Treves in [82], using a coherent states method, and that proof is given in Section 27.3. Although it is far from an elementary proof, the simplifications allowed by condition (P) permit a rather compact exposition. The last three sections of that chapter are devoted to the much more involved case of subelliptic operators satisfying condition (Ψ), and one could say that the proof is extremely complicated. Let us cite L. Hörmander in [83] :

> *For the scalar case, Egorov [84] found necessary and sufficient conditions for subellipticity with loss of δ derivatives ($\delta \in [0, 1)$) ; the proof of sufficiency was completed in [85]. The results prove that the best δ is always of the form $k/(k+1)$ where k is a positive integer...A slight modification of the presentation of [85] is given in Chapter 27 of [86], but it is still very complicated technically. Another approach which covers also systems operating on scalars has been given by Nourrigat [87, 88] (see also the book [89] by Helffer and Nourrigat), but it is also far from simple so the study of subelliptic operators may not yet be in a final form.*

Chapter XXVIII is entitled *Uniqueness for the Cauchy problem*. It appears as a natural sequel to Chapter VIII in the first book [64]. The Calderón uniqueness result along with uniqueness under a pseudoconvexity condition are given, and the notion of principal normality is enlarged, using the Fefferman-Phong inequality [62]. However pseudodifferential methods are greedy with derivatives, so that the aforementioned chapter in [64] is not entirely included in this chapter. The last section of this chapter is devoted to a result on second order operators of real principal type essentially due to N. Lerner and L. Robbiano [90].
Chapter XXIX is entitled *Spectral Asymptotics*. This chapter is devoted to the asymptotic properties of the eigenvalues and the spectral function for self-adjoint elliptic operators. If P is a positive operator of order m, $P^{1/m}$ is a pseudodifferential operator with eigenvalue λ equal to those of P for λ^m. The corresponding unitary group $e^{itP^{1/m}}$ can be viewed as a Fourier integral operator. Here also L. Hörmander presents an excellent synthesis of many works on this topic : J. Chazarain [91], J. Duistermaat and V. Guillemin [92], V. Ivrii [93], V. Guillemin [94, 95, 96], Y. Colin de Verdière [97], A. Weinstein [98].

The very last chapter is the thirtieth, *Long Range Scattering Theory*. It is devoted to the study of operators of type $P_0(D)+V(x,D)$ where P_0 is elliptic of order m and V is of order m so that $P_0(D)+V(x,D)$ is also elliptic and

$$V(x,\xi) = V_S(x,\xi) + V_L(x,\xi),$$

where the short range part V_S has coefficients decreasing as fast as an integrable function of $|x|$ and V_L satisfies some estimates similar to those satisfied by $(1+|\xi|)^m(1+|x|)^{-\varepsilon}$ for some $\varepsilon > 0$. Here also L. Hörmander gives an excellent synthesis of his work along with the works of many mathematicians, among them S. Agmon [99].

There is certainly no better conclusion to the review of this treatise than the citation for the 2006 Leroy P. Steele Prize, awarded to Lars Hörmander for mathematical exposition :

> *In these four volumes, Hörmander describes the developments [of microlocal analysis] in a treatment that is seamless and self-contained. Moreover, the effort to make this treatment self-contained has inspired him to recast, in much more simple and accessible form, the approach to much of this material as it originally appeared in the literature. An example is the theory of Fourier integral operators, which was invented by him in two seminal papers in the early 1970s. (These get a completely new and much more elegant reworking in volume four.) In brief, these four volumes are far more than a compendium of random results. They are a profound and masterful "rethinking" of the whole subject of microlocal analysis. Hörmander's four volumes on partial differential operators have influenced a whole generation of mathematicians working in the broad area of microlocal analysis and its applications. In the history of mathematics one is hard-pressed to find any comparable "expository" work that covers so much material, and with such depth and understanding, of such a broad area of mathematics.*

Intermission Mittag-Leffler 1984-1986, back to Lund 1986

L. Hörmander spent the academic years 1984-86 as director of the Mittag–Leffler Institute in Stockholm. He wrote about this :

> *I had only accepted a two year appointment with a leave of absence from Lund since I suspected that the many administrative duties there would not agree very well with me. The hunch was right…*

L. Hörmander was back at the university of Lund in the Fall of 1986.

Nonlinear hyperbolic equations

During three semesters in 1986-87, Hörmander gave some lectures on global existence or blowup for nonlinear hyperbolic equations. Ten years later, in 1996, the book *Lectures on Nonlinear Hyperbolic Differential Equations* [100] appeared in the Springer series MATHÉMATIQUES & APPLICATIONS.
Some classical topics on scalar first order equations are covered and revisited in the first chapters of the book. Chapter 5 concerns compensated compactness. The main tool is Young measures associated to an L^∞ bounded sequence of functions. The author uses them to prove "compensated compactness" theorems, generalizing the "Murat-Tartar div-curl lemma" [101], [102]. Applications of these ideas to scalar or two-by-two systems are included.
The rest of the book is devoted entirely to nonlinear problems in several space variables. The first subject which is treated is the problem of long-time existence of small solutions for nonlinear wave or Klein-Gordon equations. L. Hörmander uses the original method of S. Klainerman [103]. It relies on a weighted L^∞ Sobolev estimate for a smooth function in terms of L^2 norms of $Z^I u$, where Z^I stands for an iterate of homogeneous vector fields tangent to the wave cone. The chapter closes with a proof of global existence in 3 space dimensions, when the nonlinearity satisfies the so-called "null condition", i.e. a compatibility relation between the nonlinear terms and the wave operator.
The last part of the book is concerned with the use of microlocal analysis in the study of nonlinear equations. Chapter 9 is devoted to the study of pseudodifferential operators lying in the "bad class" $S^0_{1,1}$ (such operators are not bounded on L^2). The starting point for the study of this class is due to G. Bourdaud [104], followed by [105]. L. Hörmander proves that a necessary and sufficient condition for such an operator $a(x, D)$ to be bounded on L^2 is that the partial Fourier transform of its symbol $\hat{a}(\xi, \eta)$ satisfies a convenient vanishing property along the diagonal $\xi + \eta = 0$. These operators form a subclass of $S^0_{1,1}$ for which he discusses composition, adjoints, microlocal ellipticity and Gårding's inequality. The results of Chapter 9 are applied in Chapter 10 to construct Bony's paradifferential calculus [106, 107]. One associates to a symbol $a(x, \xi)$, with limited regularity in x, a paradifferential operator, and proves the basic theorems on symbolic calculus, as well as "Bony's paraproduct formula". Next Bony's paralinearization theorem is discussed : it asserts that if F is a smooth function and u belongs to $C^\rho(\rho > 0)$, $F(u)$ may be written as $Pu + Ru$, where P is a paradifferential operator with symbol $F'(u)$ and R is a ρ-regularizing operator. This is used to prove microlocal elliptic

regularity for solutions to nonlinear differential equations. The last chapter is devoted to propagation of microlocal singularities. After discussing propagation of singularities for solutions of linear pseudodifferential equations with symbols in the classes defined in Chapter 9, the author proves Bony's theorem on propagation of weak singularities for solutions to nonlinear equations. The proof relies on a reduction to a linear paradifferential equation, using the results of the preceding chapter.

Notions of convexity

L. Hörmander wrote in 1994 another book entitled *Notions of convexity* [108], published in the Birkhaüser Series, *Progress in Mathematics.* The main goal of the book is to expose part of the thesis of J.-M. Trépreau [109] on the sufficiency of condition (Ψ) for local solvability in the analytic category. For microdifferential operators acting on microfunctions, the necessity of condition (Ψ) for microlocal solvability was proven by M. Sato, T. Kawai and M. Kashiwara in [32]. However the book's content clearly indicates a long approach to J.-M. Trépreau's result ; the reader is invited first to a pleasant journey in the landscape of convexity and the first chapters of the book are elementary.

Students

L. Hörmander had the following Ph. D. students :

Germund DAHLQUIST, at Stockholm University, in 1958,
Vidar THOMÉE, at Stockholm University, in 1959,
Christer KISELMAN, at Stockholm University, in 1966,
Göran BJÖRCK, at Stockholm University, in 1966,
Jan BOMAN, at Stockholm University, in 1967,
Johannes SJÖSTRAND, at Lund University, in 1972,
Anders MELIN, at Lund University, in 1973,
Lars NYSTED, at Stockholm University, in 1973,
Arne ENQVIST, at Lund University, in 1974,
Gudrun GUDMUNDSDOTTIR, at Lund University, in 1975,
Anders KÄLLÉN, at Lund University, in 1979,
Nils DENCKER, at Lund University, in 1981,
Ragnar SIGURDSSON, at Lund University, in 1984,
Hans LINDBLAD, at Lund University, in 1989,
Pelle PETTERSSON at Lund University, in 1994.

Retirement in 1996

L. Hörmander retired in 1996 and became an emeritus professor. He was still very active, publishing about two or three research papers every year. His enthusiasm and interest for mathematics remained at a high level until the very end of his life.

Final comments

After this not-so-short review of Lars Hörmander's works, we see in the first place that he was instrumental in the mathematical setting of Fourier Integral Operators, (achieved in part with J. Duistermaat) and also in the elaboration of a comprehensive theory of pseudodifferential operators. Fourier Integral Operators had a long heuristic tradition, linked to Quantum Mechanics, but their mathematical theory is indeed a major lasting contribution of Lars Hörmander. He was also the first to study the now called *Hörmander's sum of squares* of vector fields and their hypoellipticity properties. These operators are important in probability theory, geometry but also gained a renewed interest in the recent studies of regularization properties for Boltzmann's equation and other non-linear equations. Hörmander played also an essential role in the completion of the theory of subelliptic operators, and there is no doubt, that without his relentless energy and talent, the clarification of this part of the theory would have probably taken many more years.
Lars Hörmander was also a great mathematical writer and a man of synthesis. The eight books published by Hörmander are reference books, all with a very personal perspective. Broad, dense and deep, these books are essentially self-contained and bring the reader up to state-of-the-art in several mathematical domains. The four-volume treatise on linear PDE, the book on several complex variables as well as the volume on non-linear hyperbolic equations are here to stay as outstanding contributions to Mathematics.

Acknowledgements. The author is grateful to J.-M. Bony and J. Sjöstrand for several helpful comments on earlier versions of this manuscript.

Bibliographie

[1] Lars Hörmander. On the theory of general partial differential operators. *Acta Math.*, 94 :161–248, 1955.

[2] Lars Gårding. Some trends and problems in linear partial differential equations. In *Proc. Internat. Congress Math. 1958*, pages 87–102. Cambridge Univ. Press, New York, 1960.

[3] Leon Ehrenpreis. Solution of some problems of division. I. Division by a polynomial of derivation. *Amer. J. Math.*, 76 :883–903, 1954.

[4] Bernard Malgrange. Equations aux dérivées partielles à coefficients constants. I. Solution élémentaire. *C. R. Acad. Sci. Paris*, 237 :1620–1622, 1953.

[5] Lars Hörmander. Local and global properties of fundamental solutions. *Math. Scand.*, 5 :27–39, 1957.

[6] Hans Lewy. An example of a smooth linear partial differential equation without solution. *Ann. of Math. (2)*, 66 :155–158, 1957.

[7] Lars Hörmander. Differential operators of principal type. *Math. Ann.*, 140 :124–146, 1960.

[8] Lars Hörmander. Differential equations without solutions. *Math. Ann.*, 140 :169–173, 1960.

[9] A.-P. Calderón. Uniqueness in the Cauchy problem for partial differential equations. *Amer. J. Math.*, 80 :16–36, 1958.

[10] Lars Hörmander. On the uniqueness of the Cauchy problem. II. *Math. Scand.*, 7 :177–190, 1959.

[11] Lars Hörmander. On the uniqueness of the Cauchy problem. *Math. Scand.*, 6 :213–225, 1958.

[12] Michael Atiyah and Daniel Iagolnitzer, editors. *Fields Medallists' lectures*, volume 5 of *World Scientific Series in 20th Century Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 1997.

[13] François Treves, Gilles Pisier, and Marc Yor. Laurent Schwartz (1915–2002). *Notices Amer. Math. Soc.*, 50(9) :1072–1084, 2003.

[14] Lars Hörmander. On local integrability of fundamental solutions. *Ark. Mat.*, 37(1) :121–140, 1999.

[15] Lars Hörmander. Autobiography of Lars Hörmander. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 82–85. World Sci. Publ., River Edge, NJ, 1997.

[16] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119 :147–171, 1967.

[17] J. J. Kohn and L. Nirenberg. An algebra of pseudo-differential operators. *Comm. Pure Appl. Math.*, 18 :269–305, 1965.

[18] André Unterberger and Juliane Bokobza. Les opérateurs de Calderon-Zygmund précisés. *C. R. Acad. Sci. Paris*, 259 :1612–1614, 1964.

[19] Lars Hörmander. Pseudo-differential operators. *Comm. Pure Appl. Math.*, 18 :501–517, 1965.

[20] Lars Hörmander. *An introduction to complex analysis in several variables*. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

[21] Lars Hörmander. L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113 :89–152, 1965.

[22] Henri Cartan. Sur les matrices holomorphes de n variables complexes. *J. Math. Pures Appl.*, 19 :1–26, 1940.

[23] Henri Cartan and Peter Thullen. Zur Theorie der Singularitäten der Funktionen mehrerer komplexen Veränderlichen. *Math. Ann.*, 106(1) :617–647, 1932.

[24] Salomon Bochner and William Ted Martin. *Several Complex Variables*. Princeton Mathematical Series, vol. 10. Princeton University Press, Princeton, N. J., 1948.

[25] Hans Lewy. On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables. *Ann. of Math. (2)*, 64 :514–522, 1956.

[26] Kiyoshi Oka. *Sur les fonctions analytiques de plusieurs variables*. Iwanami Shoten, Tokyo, 1961.

[27] Jean-Pierre Serre. Une propriété topologique des domaines de Runge. *Proc. Amer. Math. Soc.*, 6 :133–134, 1955.

[28] Andrew Browder. Cohomology of maximal ideal spaces. *Bull. Amer. Math. Soc.*, 67 :515–516, 1961.

[29] Kiyoshi Oka. Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques. *Bull. Soc. Math. France*, 78 :1–27, 1950.

[30] Yum Tong Siu. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. *Invent. Math.*, 27 :53–156, 1974.

[31] Lars Hörmander. The spectral function of an elliptic operator. *Acta Math.*, 121 :193–218, 1968.

[32] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara. Microfunctions and pseudo-differential equations. In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971 ; dedicated to the memory of André Martineau)*, pages 265–529. Lecture Notes in Math., Vol. 287. Springer, Berlin, 1973.

[33] J. Bros and D. Iagolnitzer. Tuboïdes et structure analytique des distributions. II. Support essentiel et structure analytique des distributions. In *Séminaire Goulaouic-Lions-Schwartz 1974–1975 : Équations aux dérivées partielles linéaires et non linéaires, Exp. No. 18*, page 34. Centre Math., École Polytech., Paris, 1975.

[34] Lars Hörmander. Linear differential operators. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 121–133. Gauthier-Villars, Paris, 1971.

[35] Lars Hörmander. On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseignement Math. (2)*, 17 :99–163, 1971.

[36] Lars Hörmander. Fourier integral operators. I. *Acta Math.*, 127(1-2) :79–183, 1971.

[37] J. J. Duistermaat and L. Hörmander. Fourier integral operators. II. *Acta Math.*, 128(3-4) :183–269, 1972.

[38] V.P. Maslov. Theory o] pertubations and asymptotic methods. *Moskov Gos. Univ. Moscow*, page , 1965.

[39] Ju. V. Egorov. The canonical transformations of pseudodifferential operators. *Uspehi Mat. Nauk*, 24(5 (149)) :235–236, 1969.

[40] V. I. Arnol′d. On a characteristic class entering into conditions of quantization. *Funkcional. Anal. i Priložen.*, 1 :1–14, 1967.

[41] V. S. Buslaev. The generating integral and the Maslov canonical operator in the WKB method. *Funkcional. Anal. i Priložen.*, 3(3) :17–31, 1969.

[42] Joseph B. Keller. Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems. *Ann. Physics*, 4 :180–188, 1958.

[43] L. Nirenberg and F. Treves. On local solvability of linear partial differential equations. I. Necessary conditions. *Comm. Pure Appl. Math.*, 23 :1–38, 1970.

[44] L. Nirenberg and F. Treves. On local solvability of linear partial differential equations. II. Sufficient conditions. *Comm. Pure Appl. Math.*, 23 :459–509, 1970.

[45] L. Nirenberg and F. Treves. A correction to : "On local solvability of linear partial differential equations. II. Sufficient conditions" (Comm. Pure Appl. Math. **23** (1970), 459–509). *Comm. Pure Appl. Math.*, 24(2) :279–288, 1971.

[46] L. Nirenberg and F. Treves. Solvability of a first order linear partial differential equation. *Comm. Pure Appl. Math.*, 16 :331–351, 1963.

[47] S. Mizohata. Solutions nulles et solutions non analytiques. *J. Math. Kyoto Univ.*, 1 :271–302, 1961/1962.

[48] R. D. Moyer. Local solvability in two dimensions : necessary conditions for the principal type case. Mimeographed manuscript, University of Kansas, 1978.

[49] Lars Hörmander. Pseudodifferential operators of principal type. In *Singularities in boundary value problems (Proc. NATO Adv. Study Inst., Maratea, 1980)*, volume 65 of *NATO Adv. Study Inst. Ser. C : Math. Phys. Sci.*, pages 69–96. Reidel, Dordrecht, 1981.

[50] R. Beals and C. Fefferman. On local solvability of linear partial differential equations. *Ann. of Math. (2)*, 97 :482–498, 1973.

[51] Nicolas Lerner. Sufficiency of condition (ψ) for local solvability in two dimensions. *Ann. of Math. (2)*, 128(2) :243–258, 1988.

[52] Lars Hörmander. On the solvability of pseudodifferential equations. In *Structure of solutions of differential equations (Katata/Kyoto, 1995)*, pages 183–213. World Sci. Publ., River Edge, NJ, 1996.

[53] Nicolas Lerner. Nonsolvability in L^2 for a first order operator satisfying condition (ψ). *Ann. of Math. (2)*, 139(2) :363–393, 1994.

[54] N. Dencker. The resolution of the Nirenberg-Treves conjecture. *Ann. of Math.*, 163 :2, 405–444, 2006.

[55] Lars Hörmander. Propagation of singularities and semiglobal existence theorems for (pseudo)differential operators of principal type. *Ann. of Math. (2)*, 108(3) :569–609, 1978.

[56] R. Beals and C. Fefferman. Classes of spatially inhomogeneous pseudodifferential operators. *Proc. Nat. Acad. Sci. U.S.A.*, 70 :1500–1501, 1973.

[57] R. Beals. A general calculus of pseudodifferential operators. *Duke Math. J.*, 42 :1–42, 1975.

[58] André Unterberger. Oscillateur harmonique et opérateurs pseudo-différentiels. *Ann. Inst. Fourier (Grenoble)*, 29(3) :xi, 201–221, 1979.

[59] Lars Hörmander. The Weyl calculus of pseudodifferential operators. *Comm. Pure Appl. Math.*, 32(3) :360–444, 1979.

[60] R. B. Melrose. Transformation of boundary problems. *Acta Math.*, 147(3-4) :149–236, 1981.

[61] Louis Boutet de Monvel. Comportement d'un opérateur pseudo-différentiel sur une variété à bord. I. La propriété de transmission. *J. Analyse Math.*, 17 :241–253, 1966.

[62] C. Fefferman and D. H. Phong. On positivity of pseudo-differential operators. *Proc. Nat. Acad. Sci. U.S.A.*, 75(10) :4673–4674, 1978.

[63] M. F. Atiyah and I. M. Singer. The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.*, 69 :422–433, 1963.

[64] Lars Hörmander. *Linear partial differential operators*. Die Grundlehren der mathematischen Wissenschaften, Bd. 116. Academic Press Inc., Publishers, New York, 1963.

[65] Louis Boutet de Monvel. Comportement d'un opérateur pseudo-différentiel sur une variété à bord. *C. R. Acad. Sci. Paris*, 261 :4587–4589, 1965.

[66] Gerd Grubb. Problèmes aux limites pseudo-différentiels dépendant d'un paramètre. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(12) :581–583, 1981.

[67] J. J. Duistermaat and J. Sjöstrand. A global construction for pseudo-differential operators with non-involutive characteristics. *Invent. Math.*, 20 :209–225, 1973.

[68] Anders Melin. Lower bounds for pseudo-differential operators. *Ark. Mat.*, 9 :117–140, 1971.

[69] Louis Boutet de Monvel. Hypoelliptic operators with double characteristics and related pseudo-differential operators. *Comm. Pure Appl. Math.*, 27 :585–639, 1974.

[70] Johannes Sjöstrand. Parametrices for pseudodifferential operators with multiple characteristics. *Ark. Mat.*, 12 :85–130, 1974.

[71] Louis Boutet de Monvel, Alain Grigis, and Bernard Helffer. Parametrixes d'opérateurs pseudo-différentiels à caractéristiques multiples. In *Journées : Équations aux Dérivées Partielles de Rennes (1975)*, pages 93–121. Astérisque, No. 34–35. Soc. Math. France, Paris, 1976.

[72] V. Ja. Ivriĭ and V. M. Petkov. Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations. *Uspehi Mat. Nauk*, 29(5(179)) :3–70, 1974. Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ (1901–1973), III.

[73] R. B. Melrose. Microlocal parametrices for diffractive boundary value problems. *Duke Math. J.*, 42(4) :605–635, 1975.

[74] Michael E. Taylor. Grazing rays and reflection of singularities of solutions to wave equations. *Comm. Pure Appl. Math.*, 29(1) :1–38, 1976.

[75] Gregory Eskin. Parametrix and propagation of singularities for the interior mixed hyperbolic problem. *J. Analyse Math.*, 32 :17–62, 1977.

[76] V. Ja. Ivriĭ. Wave fronts of solutions of boundary value problems for a class of symmetric hyperbolic systems. *Sibirsk. Mat. Zh.*, 21(4) :62–71, 236, 1980.

[77] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5) :593–617, 1978.

[78] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math.*, 35(2) :129–168, 1982.

[79] K. G. Andersson and R. B. Melrose. The propagation of singularities along gliding rays. *Invent. Math.*, 41(3) :197–232, 1977.

[80] James V. Ralston. Solutions of the wave equation with localized energy. *Comm. Pure Appl. Math.*, 22 :807–823, 1969.

[81] Johannes Sjöstrand. Propagation of analytic singularities for second order Dirichlet problems. *Comm. Partial Differential Equations*, 5(1) :41–93, 1980.

[82] François Trèves. A new method of proof of the subelliptic estimates. *Comm. Pure Appl. Math.*, 24 :71–115, 1971.

[83] Lars Hörmander. Looking forward from ICM 1962. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 86–103. World Sci. Publ., River Edge, NJ, 1997.

[84] Ju. V. Egorov. Subelliptic operators. *Uspehi Mat. Nauk*, 30(2(182)) :57–114, 1975.

[85] Lars Hörmander. Subelliptic operators. In *Seminar on Singularities of Solutions of Linear Partial Differential Equations (Inst. Adv. Study, Princeton, N.J., 1977/78)*, volume 91 of *Ann. of Math. Stud.*, pages 127–208. Princeton Univ. Press, Princeton, N.J., 1979.

[86] Lars Hörmander. *The analysis of linear partial differential operators. IV*, volume 275 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Fourier integral operators.

[87] J. Nourrigat. Subelliptic systems. *Comm. Partial Differential Equations*, 15(3) :341–405, 1990.

[88] J. Nourrigat. Systèmes sous-elliptiques. II. *Invent. Math.*, 104(2) :377–400, 1991.

[89] Bernard Helffer and Jean Nourrigat. *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, volume 58 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[90] Nicolas Lerner and Luc Robbiano. Unicité de Cauchy pour des opérateurs de type principal. *J. Analyse Math.*, 44 :32–66, 1984/85.

[91] J. Chazarain. Formule de Poisson pour les variétés riemanniennes. *Invent. Math.*, 24 :65–82, 1974.

[92] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic geodesics. In *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973)*, pages 205–209. Amer. Math. Soc., Providence, R. I., 1975.

[93] V. Ja. Ivriĭ. The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary. *Funktsional. Anal. i Prilozhen.*, 14(2) :25–34, 1980.

[94] Victor Guillemin. The Radon transform on Zoll surfaces. *Advances in Math.*, 22(1) :85–119, 1976.

[95] Victor Guillemin. Some classical theorems in spectral theory revisited. In *Seminar on Singularities of Solutions of Linear Partial Differential Equations (Inst. Adv. Study, Princeton, N.J., 1977/78)*, volume 91 of *Ann. of Math. Stud.*, pages 219–259. Princeton Univ. Press, Princeton, N.J., 1979.

[96] Victor Guillemin. Some spectral results for the Laplace operator with potential on the n-sphere. *Advances in Math.*, 27(3) :273–286, 1978.

[97] Yves Colin de Verdière. Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques. *Comment. Math. Helv.*, 54(3) :508–522, 1979.

[98] Alan Weinstein. Asymptotics of eigenvalue clusters for the Laplacian plus a potential. *Duke Math. J.*, 44(4) :883–892, 1977.

[99] S. Agmon. Some new results in spectral and scattering theory of differential operators on $\mathbf{R}^n$. In *Séminaire Goulaouic-Schwartz (1978/1979)*, pages Exp. No. 2, 11. École Polytech., Palaiseau, 1979.

[100] Lars Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.

[101] François Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3) :489–507, 1978.

[102] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics : Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass., 1979.

[103] S. Klainerman. The null condition and global existence to nonlinear wave equations. In *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, volume 23 of *Lectures in Appl. Math.*, pages 293–326. Amer. Math. Soc., Providence, RI, 1986.

[104] Gérard Bourdaud. Une algèbre maximale d'opérateurs pseudo-différentiels. *Comm. Partial Differential Equations*, 13(9) :1059–1083, 1988.

[105] Lars Hörmander. Pseudo-differential operators of type $1, 1$. *Comm. Partial Differential Equations*, 13(9) :1085–1111, 1988.

[106] Jean-Michel Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2) :209–246, 1981.

[107] Jean-Michel Bony. Analyse microlocale des équations aux dérivées partielles non linéaires. In *Microlocal analysis and applications (Montecatini Terme, 1989)*, volume 1495 of *Lecture Notes in Math.*, pages 1–45. Springer, Berlin, 1991.

[108] Lars Hörmander. *Notions of convexity*, volume 127 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1994.

[109] J.-M. Trépreau. Sur la résolubilité microlocale des opérateurs de type principal. In *Conference on Partial Differential Equations (Saint Jean de Monts, 1982)*, pages Conf. No. 22, 10. Soc. Math. France, Paris, 1982.