Mathematical Conversations

Hans Föllmer: Efficient Markets, Random Paths >>>

Hans Föllmer

Interview of Hans Föllmer by Y.K. Leong (matlyk@nus.edu.sg)

Hans Föllmer is renowned for fundamental contributions to statistical mechanics, stochastic analysis and mathematical finance. He is also known for his indefatigable energy and enthusiasm in actively promoting the applications of mathematics, especially to financial markets.

Having undertaken a broad education in philosophy, languages, physics and mathematics in four European universities, he obtained his doctorate (Dr. rer. Nat.) from University of Erlangen under the supervision of Konrad Jacobs. Except for a 3-year stint in the U.S. at MIT and Dartmouth College, his career was essentially cultivated to fruition in Europe — at University of Erlangen, University of Frankfurt, University of Bonn, ETH Zurich and Humboldt University in Berlin. At Bonn, he was professor twice, first at the Department of Economics and later, after a period of eleven years in Zurich, at the Department of Mathematics. Since 1994, he has been Professor of Mathematics at Humboldt University, Berlin.

His extensive publications cover several interdisciplinary areas. In addition to the influence of his pioneering research, he has made numerous contributions to the scientific communities in Europe and elsewhere through his active involvement in scientific committees and advisory boards. For his deep and wide-ranging contributions, he received the following awards: Emmy Noether award of the University of Erlangen, Science Prize of the GMÖOR (Gesellschaft für Mathematik, Õkonomie und Operations Research), Prix Gay-Lussac/Humboldt, the Georg Cantor Medal of the German Mathematical Society and an honorary doctorate from the University Paris-Dauphine.

He was also elected as member of the following scientific bodies: Academia Europaea, Deutsche Akademie der Naturforscher Leopoldina, and Berlin-Brandenburgische Akademie der Wissenschaften.

Besides giving invited lectures at major scientific meetings and universities throughout the world, he is actively engaged in the training of scientists and mathematicians both inside and outside of Europe. Among other activities, he is involved in the International Research Training Group (IRTG) Berlin-Zurich and the DFG Research Center "Mathematics for key technologies".

Since 2000, Föllmer is a regular visitor to NUS and has rendered valuable service to the Department of Mathematics and the Institute. He is a founder member of the Institute's Scientific Advisory Board (SAB) which successfully charted the direction of the Institute during its first five years. During a 3-year period in 2000-2003, he helped to develop the

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Department's new financial mathematics program, and visited the department for short periods of 4 to 8 weeks to advise and give courses on the subject.

It was during his visit to the Institute as a member of the SAB that Y.K. Leong interviewed him on behalf of *Imprints* on 4 January 2006. The following is an edited and enhanced account of this interview, in which he spoke with passion about his intellectual path from philosophy to mathematics, and gave us a rare glimpse, from the view of a pioneer at the interface of probability and finance, of the somewhat unexpected impact of stochastic analysis (an esoteric branch of mathematics) on stock markets (one of the most practical activities of an industrial society).

Imprints: Your university education seems to have been rather unusual in the sense that it was taken in many places in Germany and France. Please tell us something about it and about how you became interested in stochastic analysis.

Hans Föllmer: In the German tradition, the fact that I went to several universities is not unusual but it's quite normal and even expected traditionally. My father, for example, as a student, went to four different universities. The idea was to get to know different schools of thought in different parts of the country. In that sense, I did the normal thing. I started out in Cologne, then I went to Göttingen, and the reason that I went to France was that at that time I had already focused on one special area, and my advisor for the diploma thesis asked me to go to Paris for a year in order to learn more about it. In the meantime, he had moved from Göttingen to Erlangen, and then I joined him in Erlangen, and that was university number four - just like my father.

I: Who was your supervisor?

F: My supervisor was Konrad Jakobs. He was working in ergodic theory, and the reason that he went to Erlangen was that he wanted to establish a joint center in probability with Hans Bauer who had at the same time moved from Hamburg to Erlangen. Bauer and his students were working on the potential theory of Markov processes, and my own interest then was, in fact, closer to Bauer's than to Jakobs'. The reason I went to Paris in 1965 was that I was supposed to learn some potential theory from the sources in France — Choquet and Brelot, for example. Of course I also took other courses, and I particularly enjoyed the lectures of Laurent Schwartz and Jacques Neveu. After my year in Paris I went to Erlangen for three years. During that time there was a lot of activity in probability. Robert Blumenthal came for a year and gave a graduate course on the book on Markov processes which he was writing with Ron Getoor. There were visitors such as Paul-André Meyer, Joe Doob, Shizuo Kakutani, Alexandr Borovkov, Kiyosi Itô and Kai Lai Chung. For us graduate students that was an exciting time.

I: Were you interested in stochastic analysis right from the beginning?

F: No, I even didn't start in mathematics at the beginning. I first started to study philosophy and literature. Then I became interested in the philosophy of language, linguistics, and I thought it would be good for me to understand how formal languages like mathematics work. So I thought I would sit in on mathematics classes, and then I got interested, and slowly I got drawn into the subject. One reason was that mathematics was much better organized as a curriculum than philosophy. Philosophy was very free floating. So I got sucked into the mathematics program and started to enjoy it. There were several occasions that I thought of going back to my original interest, but I stayed on. The reason I got interested in probability rather early, in my third year of study of mathematics, may have something to do with my original motivation in philosophy because I was intrigued by the notions of probability, entropy and uncertainty. That probably played some role in my decision.

I: Was there any single person who was quite decisive in making you work in probability?

F: The reason that I decided to specialize in probability had certainly something to do with my teacher at that time, Konrad Jakobs. He is a very impressive person and has very wide interests in mathematics and beyond. I liked him a lot as a teacher, and he immediately helped me and supported me. That probably played a role, too.

I: Was it a tradition to have broad interests?

F: Yes, that was the intellectual tradition. You were encouraged to take a broad approach and I liked that. Nowadays, it is much more focused. In retrospect, it was a luxury spending time on philosophy and so on. It would be harder to do the same thing now, also in Germany, because now there is more pressure on students to proceed quickly.

I: You taught briefly for three years in the United States immediately after your doctorate. Was it a cultural or intellectual pull that made you return to Europe to establish your career in Germany?

F: That was a very difficult decision. After one year in the States, I thought, "Okay, it was all very interesting, but, no, I really want to go back to Europe." After the second year, I was no longer so sure, and in the third year I was strongly tempted to stay. Clearly the scientific situation in the United

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States was very attractive. But I was already married and we had a child, and finally we decided to go back to Germany, mainly for cultural reasons. Soon after, I had another option to go back to North America, but I also had attractive offers in Germany, and so we decided to stay. But it was more the cultural pull, not so much the intellectual in the professional sense.

I: Your research work was initially in stochastic processes, which is theoretical probability theory, and you soon began to work on stochastic problems in other fields. Why mathematical finance and not other areas like biology?

F: My research was primarily in stochastic processes for 30 years, not only initially. But I had one first contact with mathematical finance already in 1971 when I was at Dartmouth. At that time, I had an undergraduate student who wanted to write a senior thesis in probability, and he proposed to work on an optimal stopping problem related to insider information in finance. This was David Kreps, who went on to become professor of economics at Harvard and Stanford and to receive the Clark medal in 1989, and who is now dean of the business school at Stanford. I learned from him what an option is. That was my first contact with mathematical finance. But for a long time I continued to work in probability, on problems in martingale theory, in interacting particle systems, and in stochastic analysis. For several years, I worked on questions motivated by the interface between probability and statistical mechanics, especially probabilistic approaches to phase transitions, Gibbs measures, and large deviations. My interest in mathematical finance became more systematic only much later, in the mid-80s. Actually, it was again triggered off by David Kreps. David spent a sabbatical in Cambridge and he came over to ETH Zurich, where I was teaching at that time, and gave a seminar related to the Black-Scholes pricing formula for options. I got intrigued and started to think about it. Then Dieter Sondermann, a colleague from Germany, visited ETH Zurich for a month. At the same time, he was doing consulting work with a major Swiss bank, and we started to work together on some mathematical aspects of option pricing. From that time on, I took a more systematic interest.

I: Did you work on a specific problem with this colleague of yours?

F: Yes, we looked at the problem of hedging financial derivatives in situations where the Black-Scholes paradigm of a perfect hedge breaks down, and we used arguments from martingale theory. By the way, Dieter Sondermann was professor of statistics in the economics department at the University of Bonn. He was holding the same position that I had held from 1974 to 1977 before I went to ETH Zurich. At that time, I had a position as professor of statistics at the economics department of the University of Bonn. That was from 1974 to 1977. In 1974, I had three options — two offers for positions in mathematics and one from the economics department. At that time, I decided to take the economics offer because I wanted to learn what those guys were doing. The experience of three years in the economics department was probably responsible for my later decision to pursue questions in mathematical finance. After three years, however, I had an offer from ETH Zurich and I thought it was a good time to go back to mathematics. One aspect of this was that in doing research with students on questions which I liked, the conditions were better in the mathematics department than in the economics department. But I never regretted the decision to go to the economics department for some time because it was a very enriching experience to get to know this other culture. At that time, Gerard Debreu (who later received the Nobel Prize in economics for work in microeconomic equilibrium theory) was visiting the economics department in Bonn for a year to work with Werner Hildenbrand. He came with a strong group of young economists from Berkeley which included Truman Bewley (later at Yale), Mukul Majumdar (later at Cornell), Alan Kirman (later at Marseille) and Andreu Mas-Collel (for a long time at Harvard before he became minister of universities and research in Catalunya). That was a very stimulating environment, and I enjoyed that a lot.

I: How much of the field of mathematical finance has been accepted as an integral part of economics?

F: The fact that some of the Nobel prizes have been awarded to work in quantitative and even mathematical finance shows that the field has a lot of acceptance within the community of economists. I was more concerned with the other side — how well accepted is mathematical finance as a part of mathematics? My main interest was always in questions which are motivated by the financial applications, but which also have some intrinsic mathematical interest and can be treated as research problems in their own right from the mathematical point of view.

I: Decades ago, the general public would associate financial mathematics with more commercial activities like accounting and book-keeping. Do you think that this general public perception has been significantly raised to a higher level?

F: Several decades ago, before the early 70s, I would have had the same perception. Since then there has been really a spectacular change and a dramatic increase in mathematical sophistication. Mathematical finance has become a new source of appreciation and esteem for mathematics in the

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eyes of the general public. In the financial industry, the number of professional mathematicians working there has become much higher than what it used to be. I think that has generated a lot of respect for mathematics within that community and also in a wider public. When the Nobel Prize was given to Mertens and Scholes for their famous option pricing formula, this was one of the rare occasions where a mathematical formula appeared on page 1 of the *New York Times*. Yes, public perception of mathematics has been significantly raised.

I: If I'm not mistaken, some kind of empirical stochastic studies of the stock market were actually carried out before probability theory was rigorously established. We know that sophisticated mathematical tools are now used to deal with problems of the stock market. Have those problems also contributed to and possibly influenced the theoretical development of probability theory? If so, could you give us some examples?

F: I think there is an interplay between direct concerns with the stock market and the development of probabilistic concepts and methods. One very basic mathematical object in probability theory is Brownian motion, which plays a fundamental role for a number of reasons. Brownian motion was proposed (not under that name) in 1900 by Bachelier in his thesis with Poincaré in Paris as a model for price fluctuations in the stock market. Thus the aim to describe price fluctuation in mathematical terms has motivated a very important step in the development of the theory of stochastic processes. From then on, the original financial input to the theory of Brownian motion was for a long time forgotten. The theory of Brownian motion was developed on its own for intrinsic mathematical reasons and it was only in the 60s that the original work of Bachelier was taken seriously again from the financial point of view. The group of Paul Samuelson at MIT started to use it systematically in the mid-sixties, and since then Brownian motion (on the logarithmic scale) serves as a benchmark model in finance.

I: Was it at a rigorous level?

F: The original work of Bachelier contained a number of important ideas. From the modern point of view, it was not as rigorous as what you would like to see nowadays. The fundamental mathematical problem of constructing Brownian motion rigorously as a measure on the space of continuous paths was only solved 23 years later by Norbert Wiener. But on the more qualitative level, some very important ideas, for example the reflection principle for Brownian motion, already appeared in Bachelier's work. It also contained a formula for option pricing. It's not the one which later became the canonical pricing formula because it was based not on a logarithmic Brownian motion but on the original Brownian motion itself, and one crucial argument for the Black-Scholes formula was missing, namely the construction of a perfect hedge.

You asked whether those problems contributed to and possibly influenced the theoretical development of probability. My answer would be "yes". I have already given the first example. The introduction of Brownian motion was motivated by the financial interpretation. Another example is the revival of martingale theory in the late 80s. Martingale theory had flourished in the 60s and 70s. The financial interpretation suddenly provided a fresh look and new questions. Several theoretical developments are due to that financial interpretation. One example is the pricing theory in incomplete financial markets. Let me explain. From the mathematical point of view, the Black-Scholes formula simply reduces to the following basic fact about non-linear functionals of Brownian motion. A fundamental theorem of Kiyosi Itô says that such a functional can be represented as a stochastic integral of Brownian motion. In the financial interpretation, the integrand can be interpreted as a trading strategy. The non-linear functional describes the payoff of a financial derivative, for example a call option. Thus Itô's representation theorem shows how to represent the payoff as a result of a trading strategy involving the underlying financial assets. This leads to a recipe for pricing. The initial constant which generates, using the trading strategy, the payoff may be viewed as the cost of replicating the financial derivative. This implies that the initial cost is the right price for that option. Otherwise there would be an arbitrage opportunity. That is the key to what is known as the Black-Scholes formula. From the mathematical point of view, one could say that it is simply an application of a basic representation theorem in stochastic analysis for functionals of Brownian motion.

I: Who was the first to make this observation?

F: Originally, the Black-Scholes formula was not derived by a representation theorem. It was derived by a direct argument using the Itô calculus and the solution of an appropriate partial differential equation. The full power of the representation theorem is needed if you pass from simple financial derivatives such as call options to more exotic options. Then you need the functional on the full path space. The connection to the representation theorem was clarified by David Kreps, whom I've mentioned earlier, Michael Harrison and Stan Pliska in the 80s. They recognized the relevance of previous work on the representation problem which had been done in martingale theory. It is known that

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the representation theorem holds if and only if there is a unique martingale measure. How to explain a martingale measure? Typically one fixes a probabilistic model for the price process, for example, a geometric Brownian motion. Such a model is specified by a probability measure on path space. If you now change the model by switching to another probability measure which is equivalent to the original one such that the given process becomes a martingale under that new measure, then that measure is called a martingale measure. To be a martingale means to behave like a fair game with respect to that measure. This notion of a martingale measure is very fundamental in mathematical finance. That the representation theorem holds is equivalent to uniqueness of the equivalent martingale measure. This had been shown, quite independently of the financial interpretation, already in the 70s and early 80s in the French school of probability; in particular by Jean Jacod and Marc Yor.

New questions which arose had to do with the fact that the martingale measure may not be unique. Then the situation becomes more complicated. The question arises: which martingale measure should one choose as the pricing mechanism. How should one construct a reasonable hedging strategy? This question leads to a projection problem for martingales and, more generally, for semi-martingales. It triggered off a new development in probability theory where the projection theory of Kunita-Watanabe for martingales was extended to semi-martingales. So that was a new version of a basic projection problem in probability which was motivated by finance.

Another example is the following. If you want to hedge the financial derivative, you may insist on staying on the safe side and make sure that there is no shortfall at the end of the day. Mathematically, this leads to the theory of super-hedging which can be seen as a new generalization of the classical Doob-Meyer decomposition for supermartingales, a fundamental theorem in martingale theory. This new version, now often called the optional decomposition theorem, was developed first, in a special context, by Nicole El Karoui and then in full generality by Dima Kramkov, a former student of Albert Shiryaev in Moscow, at that time a postdoc in Bonn, and now professor at Carnegie-Mellon. For this work he received a prize of the European Mathematical Society for junior mathematicians in 1996. This is another example where a question in finance led to a new problem in probability and triggered off a significant advance on the theoretical level.

I: The concepts and ideas are totally new?

F: The optional decomposition is definitely a new step. It is not a straightforward generalization. You can see that in discrete time. There the Doob-Meyer decomposition can be written down in three lines, but the extension to the optional decomposition, even in discrete time, takes several pages. It involves a new combination of martingale arguments and arguments from convex analysis. It is not just a technical refinement; it is a conceptual advance.

I will give you a third example. In applying arguments from mathematical finance, you usually fix a probabilistic model. Typically, there is a significant amount of model uncertainty. You cannot be sure that the chosen probability measure really describes the objective situation. One way of dealing with that is to take into account a whole class of possible probability measures. Then many new problems arise. For example, the classical problem of optimal portfolio choice translates into a new projection problem. You have to project the whole class of model measures on the class of martingale measures. In the usual case, you would simply project one single measure on a given convex class of measures. This problem is well understood, especially if the projection problem is formulated in terms of relative entropy. The question of model uncertainty leads to a new robust version of the classical projection problem, which has been treated only recently. It has been solved last year in joint work with Anne Gundel while we were both at the IMA in Minneapolis for a program in financial engineering.

I: Does the computer play a significant role in your work on stochastic finance? Do you rely on the empirical data to shape your ideas?

F: I am staying on the theoretical side. I do not work myself on the computer or use simulations, but I follow some of the developments on the empirical side. Some of my own work is motivated by empirical work on the microstructure of financial time series. There are new modeling issues which arise. For example, if you look at financial data on a tick by tick basis, it provides the motivation to model the dynamics of an order book. So you do not immediately switch to the mesoscopic level of description by means of stochastic differential equations but you try to model the dynamics of the market microstructure. That also raises the question of how do you model in mathematical terms the interaction of many agents who trade and place their orders. To develop mathematical models for the microstructure of financial markets is a very challenging research program which calls for methods developed in the theory of interacting particle systems. I've recently been involved in some related issues in joint work with Ulrich Horst, a former PhD student in Berlin who is now at UBC in Vancouver, and with Alan Kirman, whom I have already mentioned before.

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I: It seems that mathematical finance is built on axiomatic and abstract principles (like the efficient market principle). Have these principles been tested and verified?

F: The efficient market hypothesis comes in different forms. In its strong form, it says that the price fluctuation you observe behaves like a martingale. As a special case, it would be the random walk hypothesis, which assumes that price moves like a random walk. What you see is usually not so far from the martingale property, but there is a lot of evidence that you should not take it literally. In fact, that form of the hypothesis is too strong. If you move away from that hypothesis, it means that due to a basic systems theorem of Doob there are strategies that generate a positive expected gain. But there is a weaker form of the hypothesis which is much more flexible. It says the following. There may be strategies with positive expected gain but it is not possible to have positive expected gain and zero downside risk. In other words, there are no free lunches. That makes economic sense because if free lunches were available, there are enough clever people around to seize the opportunity and to wipe them out. In this more flexible form, the hypothesis is widely accepted. There is a broad consensus that you don't find free lunches even though you may be able to make profits with positive expected gain accepting some downside risk. In this weaker form, the hypothesis has been a rich source of interesting mathematical developments. It has been shown that the absence of arbitrage opportunities is mathematically equivalent to the fact that there are equivalent martingale measures. That is an existence theorem. Modern mathematical finance starts on that basis.

I: Can mathematical finance be considered a science?

F: If you translate the question into German, the answer would be clearly "yes". In German, "science" is "Wissenschaft" and "Wissenschaft" is rather broad. It's not just natural science, it also includes the social sciences, economics, and finance as well.

I: If it is a science, one should be able to falsify principles or hypotheses in mathematical finance.

F: Yes. For example, there is a lot of empirical evidence that the efficient market principle in its strong form does not hold. Personally, I do not work under that strong hypothesis. I do work under the weaker one. There is no significant evidence that I know of which would refute it.

I: Does it mean that in finance there are such things as laws that govern the behavior of stock markets?

F: I do believe in the relevance of probabilistic laws in finance. It's reasonable to describe price fluctuations in terms of probability measures on certain path spaces. The absence of free lunches amounts to the existence of an equivalent martingale measure, and this implies that continuous price fluctuations are nowhere differentiable. This can be viewed as a law which explains the erratic price behavior of a liquid stock which you actually see on a mesoscopic time scale. If you take the problem of pricing financial derivatives, you can show that a price must satisfy certain bounds if it does not create arbitrage opportunities. Such arbitrage bounds can be seen as a law, too. Mathematical finance is certainly a science, by my understanding of science.

I: Have you done consultation work for any financial organization?

F: No, I have not done that personally. But some of my co-authors have been involved in that. I have former students who are involved in that. I am following some of their activities, but I try not to get involved myself.

I: It's very lucrative.

F: It may be lucrative, but it also may change your life. I had occasion to watch while working on a joint paper how my co-author was, every once in a while, called to the phone because the program had to be urgently modified in some bank where his ideas were being implemented. I would not like that kind of pressure.

I: What is your advice to graduate students who are keen on a career in mathematical finance?

F: My advice to my own students is to get a broad and solid education in mathematics and not to specialize too early. Even if you decide to work in this area of finance, either in academia or the financial industry, it's a field that evolves rather fast. You need a lot of flexibility, also on a mathematical level. It's not clear that it will be enough to know the tools, for example, needed to understand the Black-Scholes formula. Other challenges may come up which may require very different techniques. I already gave you one example — the microstructure of financial markets. You have to be a good probabilist to react efficiently and to use other methods as well. To my own students, I recommend them not to narrow down too early but to make sure that they are comfortable with a wide range of techniques in probability and analysis.